GENERAL 3-D STRESS SYSTEM

Right-handed set of orthogonal axes labeled 1,2,3

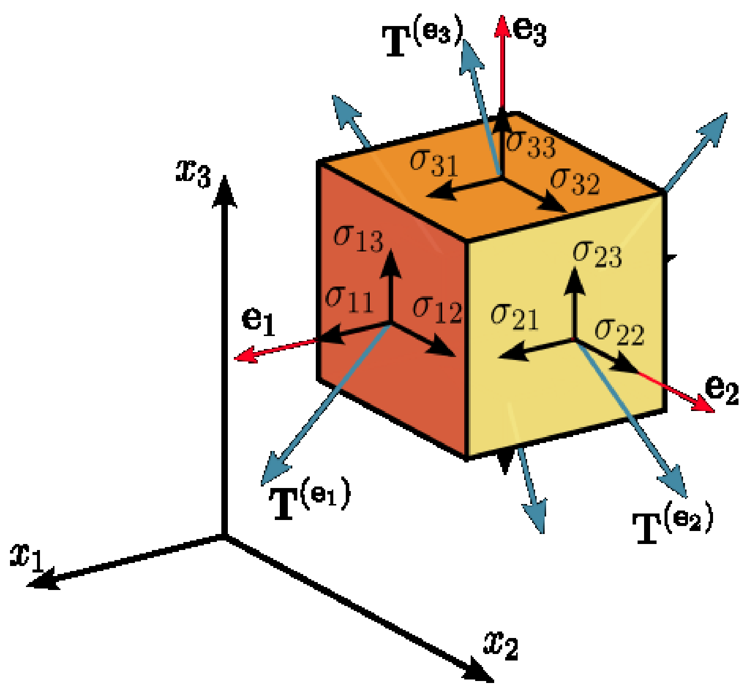
Stress components σ_{ij} where i = 1,2,3 and j = 1,2,3

i represents the direction of the normal to the plane on which a stress component acts

j represents the direction in which the stress component acts

A component with i = j is a normal stress.

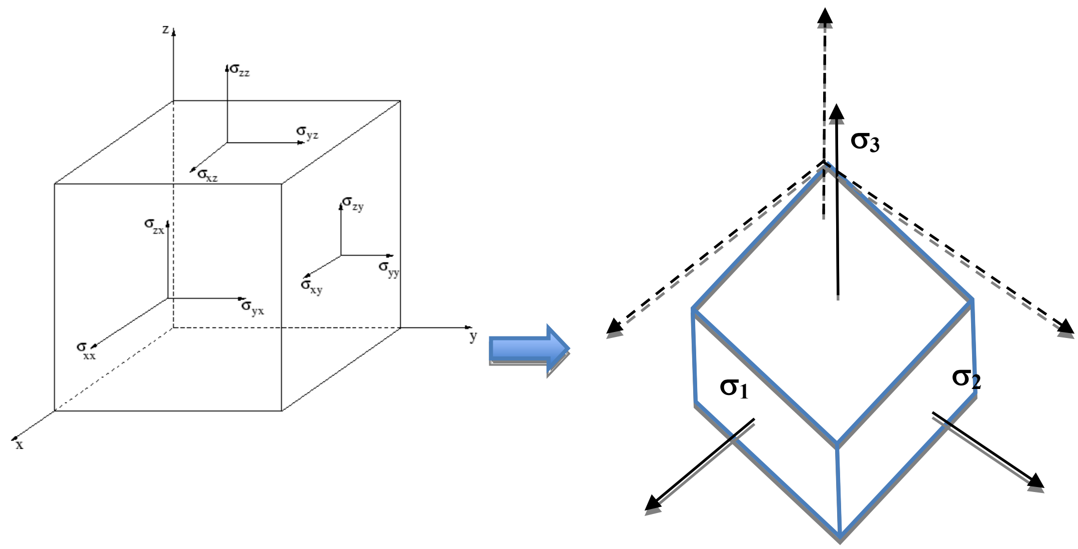
A component with i ≠ j is a shear stress.



The stress tensor is $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

For static equilibrium, $\sigma_{21} = \sigma_{12}$, $\sigma_{31} = \sigma_{13}$, and $\sigma_{32} = \sigma_{23}$. Stress tensor is symmetrical about its leading diagonal.

The test cube can always be rotated into a particular orientation where all shear components vanish:



Then tensile components σ_{11}, σ_{22}, and σ_{33} become principal stresses σ_1, σ_2, and σ_3.
The principal stress tensor is:

$$\sigma_p = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Transformation of Stresses

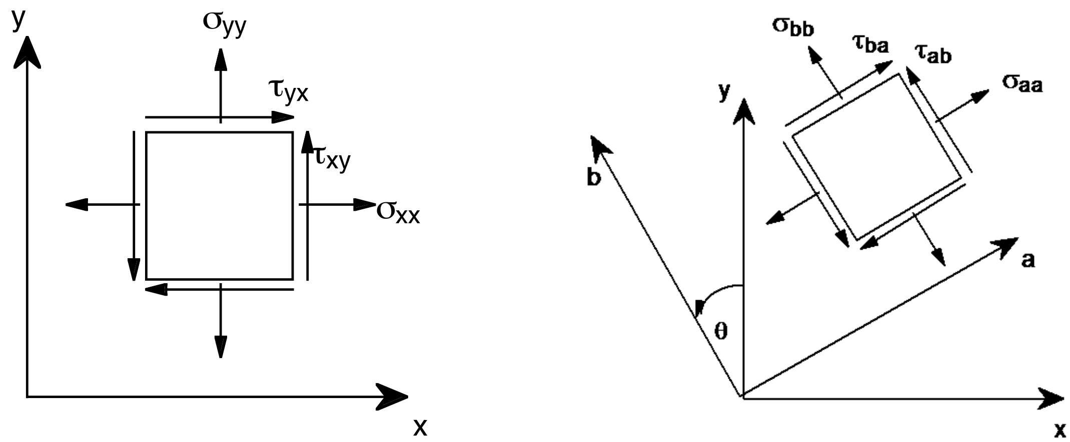
Stresses depend on the choice of coordinate axes. If we rotate the axes then although the internal distribution of forces will not change, the magnitude of the stresses with respect to the new coordinate system will vary.

Tensor transformation rule: $\sigma_p = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} l_{i'i} l_{j'j}$ where $l_{i'i} = \cos\theta_{i'i}$ - directional cosine.

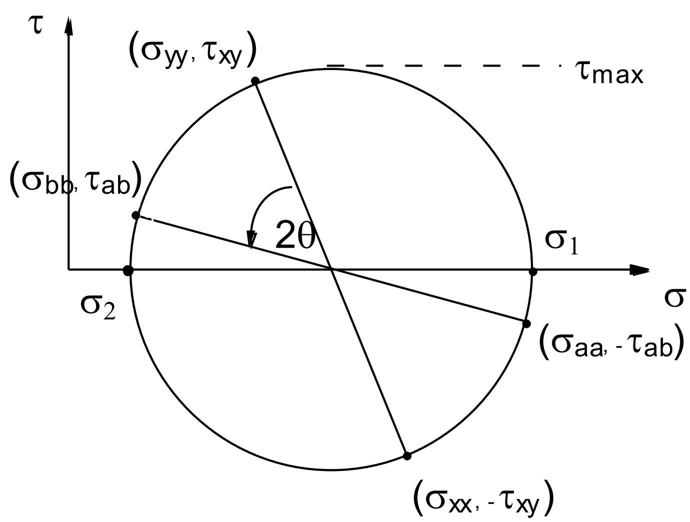
So, $\sigma_{1'1'} = \sigma_{11} l_{1'1} l_{1'1} + \sigma_{12} l_{1'1} l_{1'2} + \sigma_{13} l_{1'1} l_{1'3} + \ldots$ (6 more terms!)

Alternatively, we can calculate the normal and shear stresses for any inclined plane of interest directly (take notes from here).

All failure criteria for materials involve either principal stresses or maximum shear stresses



Mohr's circle for stress



Center of circle: $\left(\sigma_{xx}+\sigma_{yy}\right)/2$

Radius of circle: $\frac{1}{2}\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\,\tau_{xy}^2}$

Principal normal stresses: $\sigma_1, \sigma_2 = \frac{\left(\sigma_{xx}+\sigma_{yy}\right)}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\,\tau_{xy}^2}$

(in direction for which $\tau_{12} = 0$)

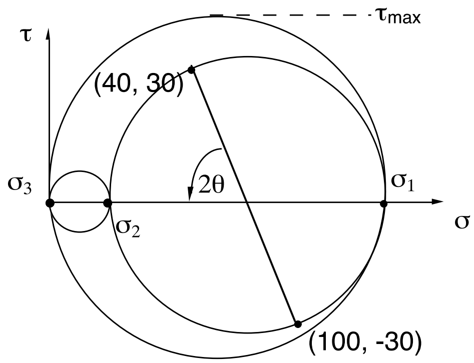
Maximum shear stresses inclined at 45^o ($2\theta = 90^o$) to principal axes

"Principal" or "maximum" shear stress in *x-y* plane:

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

Example $\sigma_x = 100$, $\sigma_y = 40$, $\sigma_z = 0.0$, $\tau_{xy} = 30$, $\tau_{xz} = \tau_{yz} = 0.0$ MPa

Since $\tau_{xz} = \tau_{yz} = 0.0$ MPa, $\sigma_{zz} = \sigma_3 = 0$ MPa



From before: $\sigma_1 = 112.4$, $\sigma_2 = 27.6$ MPa

Radius 1-2: $\tau_{12}\big|_{max} = 42$ MPa (as before)

Radius 2-3: $\tau_{23}\big|_{max} = (27.6 - 0.0)/\ 2 = 14$ MPa

Radius 1-3: $\tau_{13}\big|_{max} = (112.4 - 0)/\ 2 = 56$ MPa

$\therefore$ Largest shear stress is 56 MPa in 1-3 plane (not *x-y* plane)