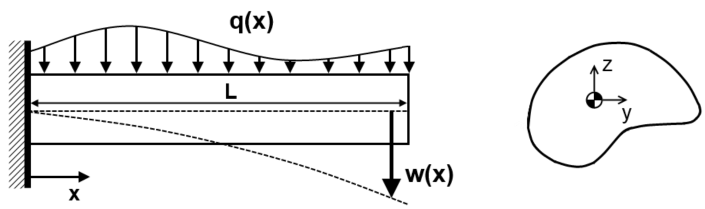
# Cellular Solid Mechanics

## Mechanics of Beams:

- Euler Bernoulli Beam Theory
  - Simplified relationship relating deflection $w(x)$ to applied transverse load $q(x)$
  - Applies well for slender beams: $\frac{EI}{\kappa L^2 AG} \ll 1$ – or – $\frac{r}{L} < 10$, with Young's modulus ($E$), area moment of inertia ($I$), cross sectional area ($A$), beam length ($L$), shear modulus ($G$)




*Beam under applied load q(x) with corresponding cross section*

  - Governing ODE:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x)$$

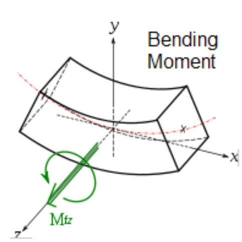
  - Shear Force:

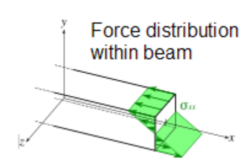
$$Q = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$$

  - Bending Moment:

$$M = -EI\frac{d^2w}{dx^2}$$

  - Area moment of inertia:
    - $I = I_y = \iint z^2 dy\, dz$
    - Circular beam with radius $r$: $I_y = \frac{\pi}{4}r^4$
    - Square beam with width $b$ and height $h$: $I_y = \frac{bh^3}{12}$

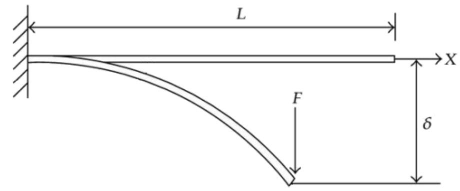


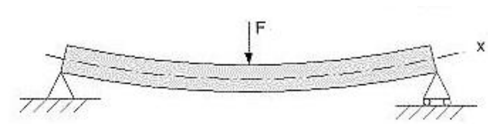

  - Stress
    - Varies linearly in the cross section of the beam
    - $\sigma = 0$ in the neutral axis of the beam

$$\sigma = \frac{Mz}{I}$$

  - Cantilever beam solution
    - BC's: $w(0) = 0, w'(0) = 0, w''(L) = 0, w'''(L) = const.$




- $w_{max} = w(L) = \frac{FL^3}{3EI}$
- $\sigma_{max} = \sigma(0) = \frac{FLz_{max}}{I}$

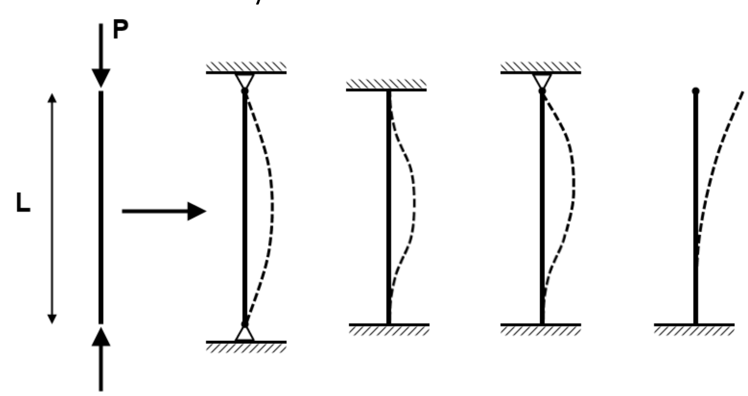



- Solution to simply supported beam
  - BC's: $w(0) = w(L) = 0, w''(0) = w''(L) = 0$
  - $w_{max} = w\left(\frac{L}{2}\right) = \frac{FL^3}{48EI}$
  - $\sigma_{max} = \sigma\left(\frac{L}{2}\right) = \frac{FLz_{max}}{4I}$
- General solution
  - $w(x) = \frac{1}{EI}\left(\frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4 + \int\int\int\int q(x)dx^4\right)$

- <u>Euler Buckling</u>
  - Elastic instability causing a bifurcation to a lower energy bent state
  - Solution to the ODE: $EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0$
  - General solution is: $w(x) = A\sin(kx) + B\cos(kx) + Cx + D$, where $k = \sqrt{\frac{P}{EI}}$
  - The boundary conditions are then used to determine the post-buckled shape.
  - The critical load at the lowest mode buckled state can be found to be:

$$P_{cr} = \frac{\pi^2 EI}{(kL)^2}$$

  - $k$ is an "effective length factor"
    - $k = 1$ for pinned-pinned boundary
    - $k = 0.5$ for fixed-fixed boundary
    - $k = 0.699$ for fixed-pinned boundary
    - $k = 2$ for fixed-free boundary




*Post-buckled states of different boundary condition beams*

- <u>Yielding failure in beams</u>
  - When the axial stress in a beam reaches the yield stress (tension, compression, or bending), it will begin to yield.

**Rigidity Theory:**

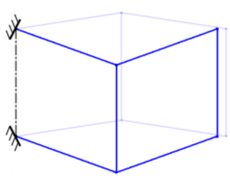
- How do we determine if a pin-jointed structure is rigid?
- "Rigid" means that any deformation of the structure requires an increase in strain energy.
- <u>Maxwell's Equation</u>
  - Consider a structure with $j$ joints and $b$ bars subject to $k$ kinematic constraints
  - We can say the structure can be rigid if it satisfies the equation:

$$dj - b - k \leq 0 \begin{cases} d = 2 \text{ in 2D} \\ d = 3 \text{ in 3D} \end{cases}$$

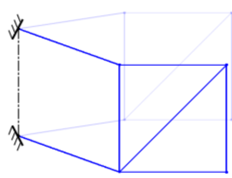
  - This is a necessary but not sufficient condition.
  - The equation can be generalized to:

$$dj - b - k = m - s$$

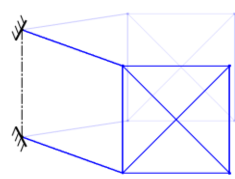
  - Here, $m$ represents the number of mechanisms and $s$ represents the states of self-stress
  - A mechanism, or inextensional mechanism, means the structure can be moved without the application of stress in the bars.
  - A self-stress means there can be an applied stress in the bars without any corresponding motion of the structure. It also means the structure is 'statically indeterminate'.
  - Examples:
    - Squares on a hinge



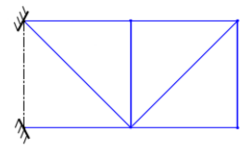
b = 2, j = 6, k = 4
2j - b - k = 2
m = 2, s = 0



b = 7, j = 6, k = 4
2j - b - k = 1
m = 1, s = 0

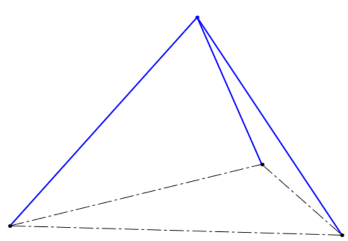


b = 8, j = 6, k = 4
2j - b - k = 0
m = 1, s = 1

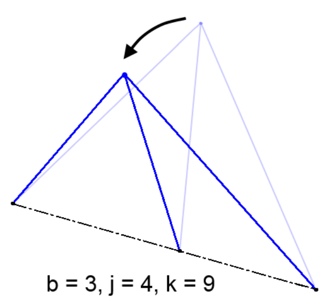


b = 8, j = 6, k = 4
2j - b - k = 0
m = 0, s = 0

    - 3-bar structure



b = 3, j = 4, k = 9
3j - b - k = 0
m = 0, s = 0



b = 3, j = 4, k = 9
3j - b - k = 0
m = 1, s = 1

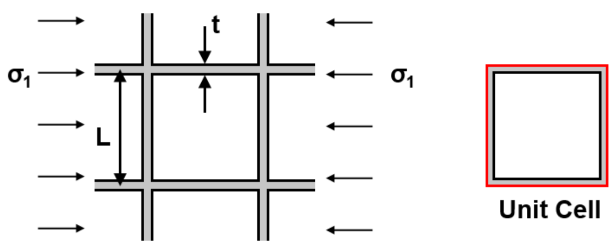
- Equilibrium Matrix Method
  - "Matrix Analysis of Statically and Kinematically Indeterminate Frameworks", S. Pellegrino & C.R. Calladine (1986)
  - Create a system of equations relating the force at the nodes $\boldsymbol{f}$ to the uniaxial force in the beams $\boldsymbol{p}$ with an equilibrium matrix $\boldsymbol{A}$.

$$\boldsymbol{f} = \boldsymbol{A}\boldsymbol{p}$$

  - $\boldsymbol{f}$ is a vector of length $3j - k$ (3 for each dimension in 3D and $k$ kinematic constraints)
  - $\boldsymbol{p}$ is a vector of length $b$
  - $\boldsymbol{A}$ is a matrix of size $b$ x $(3j - k)$
  - The equilibrium matrix can be used to determine the number of inextensional mechanisms and states of self-stress.
  - $m = b - rank(\boldsymbol{A})$
  - $s = (3j - k) - rank(\boldsymbol{A}^T)$
  - A singular value decomposition (SVD) can be performed on the matrix to find the inextensional mechanisms of the structure.

## Mechanics of 2D Structures:

- Square Lattice
  - Uniaxial compression of all the beams when loaded in the $x_1$ and $x_2$ directions.




*Square lattice in compression*

  - The structural stiffness can be found using the rule of mixtures to be:

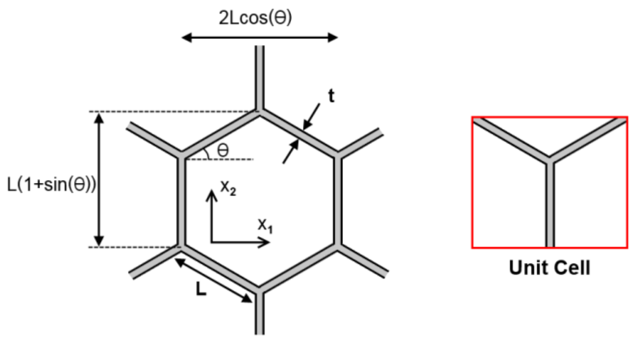
$$E_1 = \frac{t}{L} E_s$$

  - The density of a square lattice is $\bar{\rho} = \frac{2btL}{bL^2} = \frac{2t}{L}$
  - Plugging this in, we get

$$\bar{E} = \frac{E_1}{E_s} = \frac{1}{2}\bar{\rho}$$

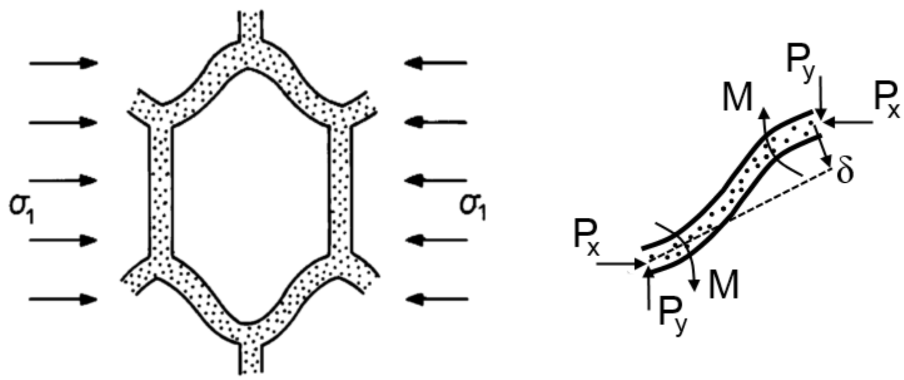
  - Linear scaling of strength and stiffness with density.
  - Highly sensitive to imperfections.

- Honeycomb
  - Bending dominated structure (2D lattice with depth $b$)




*Hexagonal Lattice*

  - The strength and stiffness in uniaxial compression are governed by bending of the beams.
  - Analysis for uniaxial compression in x-direction:




*Uniaxial compression in the $x_1$ direction*

  - The load $P$ on the unit cell that arises from the stress is $P = \sigma_1 L(1 + \sin(\theta))b$
  - The bending moment that arises in the beam can be found to be:

$$M = \frac{PL\sin(\theta)}{2}$$

  - From beam theory, the deflection is then:

$$\delta = \frac{PL^3\sin(\theta)}{12E_s I}$$

  - The moment of inertia of a beam is $I = bt^3/12$
  - The deflection of the beam in the $x_1$ direction is $\delta\sin(\theta)$
  - From this, the strain can be found to be:

$$\varepsilon_1 = \frac{\delta\sin(\theta)}{L\cos(\theta)} = \frac{PL^2\sin^2(\theta)}{12E_s I\cos(\theta)} = \frac{\sigma_1\sin^2(\theta)\,(1+\sin(\theta))}{E_s\cos(\theta)}\left(\frac{L}{t}\right)^3$$

- The Young's modulus for the structure is defined as $E_1 = \sigma_1/\varepsilon_1$

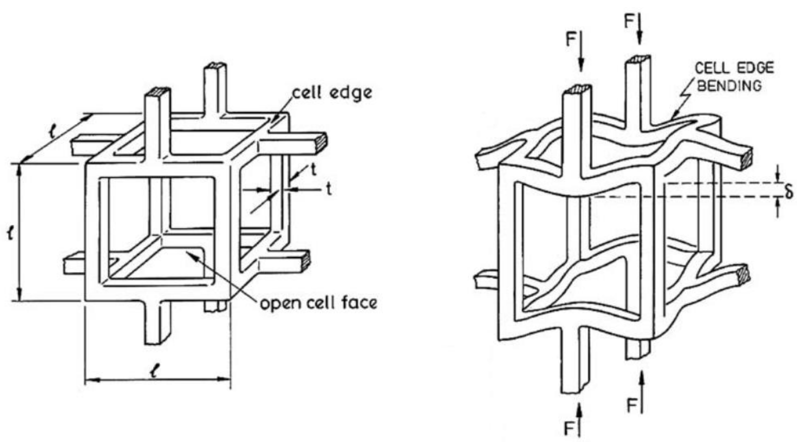
$$\bar{E} = \frac{E_1}{E_s} = \frac{\cos(\theta)}{(1+\sin(\theta))\sin^2(\theta)}\left(\frac{t}{L}\right)^3 = \frac{4\sqrt{3}}{3}\left(\frac{t}{L}\right)^3$$

- The relative density of a hexagon is: $\bar{\rho} = \frac{2}{\sqrt{3}}\left(\frac{t}{L}\right)$
- Plugging this in, we get:

$$\boxed{\bar{E} = \frac{3}{2}\bar{\rho}^3}$$

**Mechanics of 3D Structures:**

- <u>Open Cell Foam Model</u>




*Open cell foam unstressed and under an applied load*

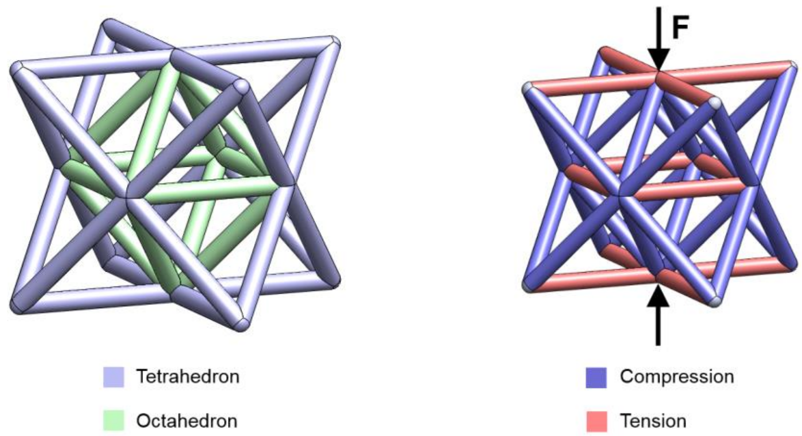
- The density of an open cell foam scales as $\bar{\rho} \propto \left(\frac{t}{L}\right)^2$
- The area moment inertia of a square beam scales as $I \propto t^4$
- Because we have a beam in bending, the deflection scales as $\delta \propto \frac{FL^3}{E_s I}$
- Stress scales with applied load as $\sigma \propto \frac{F}{L^2}$
- Strain scales with deflection as $\varepsilon \propto \frac{\delta}{L}$
- Plugging this in for stiffness, we get

$$E_1 = \frac{\sigma}{\varepsilon} = C\frac{E_s I}{L^4} = CE_s\left(\frac{t}{L}\right)^4$$

- Using our constituent relationship for relative density, we can say

$$\boxed{\bar{E} = \frac{E_1}{E_s} = C_1\bar{\rho}^2}$$

- Octet-truss (stretching-dominated solid)




*Octet-truss structure*

- The octet-truss is a fully stretching dominated 3D structure, meaning there are no inextensional mechanisms.
- In small strain compression, it is assumed that the beams perpendicular to the applied load carry the stress in tension and allow for deflection of the structure.
- Because it is a uniaxial load that causes the deflection, the stress will scale linearly with relative density, similar to the square lattice and triangular lattice cases.

$$E = 0.3 E_s \bar{\rho}$$

$$\sigma_y = 0.3 \sigma_{ys} \bar{\rho}$$

- The $0.3$ arises because only $\sim 1/3$ of the structure (the beams in tension) contributes to the global deflection.
- See: "Effective properties of the octet-truss lattice material", V.S. Deshpande, N.A. Fleck, M.F. Ashby (2001)

## Property Scaling with Relative Density:

- General scaling of strength, stiffness, and fracture toughness with relative density can be defined as

$$E = BE_s\bar{\rho}^b$$

$$\sigma_y = C\sigma_{ys}\bar{\rho}^c$$

$$K_{IC} = D\sigma_{TS}\bar{\rho}^d\sqrt{L}$$

- In 2D, these relations can be defined for different geometries as:

| **Geometry** | $B$ | $b$ | $C$ | $c$ | $D$ | $d$ |
|---|---|---|---|---|---|---|
| Hexagonal | 3/2 | 3 | 1/3 | 2 | 0.90 | 2 |
| Triangular | 1/3 | 1 | 1/3 | 1 | 0.61 | 1 |
| Kagome | 1/3 | 1 | 1/2 | 1 | 0.21 | 1/2 |

- In 3D, we instead define scaling as a function of topology and whether the structure is stretching dominated or bending dominated.

| **Topology** | $B$ | $b$ | $C$ | $c$ |
|---|---|---|---|---|
| Bending-dominated | 1 | 2 | 0.3 | 3/2 |
| Stretching-dominated | 0.3 | 1 | 0.3 | 1 |
| Stochastic | 1 | 3 | 1 | 2 |