Cellular Solid Mechanics

Mechanics of Beams:

- **Euler Bernoulli Beam Theory**
  - Simplified relationship relating deflection \( w(x) \) to applied transverse load \( q(x) \)
  - Applies well for slender beams: \( \frac{EI}{kL^2AG} \ll 1 \) or \( \frac{r}{L} \ll 1 \), with Young’s modulus \( E \), area moment of inertia \( I \), cross sectional area \( A \), beam length \( L \), shear modulus \( G \)

\[ Beam\ under\ applied\ load\ q(x)\ with\ corresponding\ cross\ section \]

  - Governing ODE:
    \[
    \frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) = q(x)
    \]
  - Shear Force:
    \[
    Q = -\frac{d}{dx} \left( EI \frac{d^2w}{dx^2} \right)
    \]
  - Bending Moment:
    \[
    M = -EI \frac{d^2w}{dx^2}
    \]
  - Area moment of inertia:
    - \( I = I_y = \int \int z^2 dy \, dz \)
    - Circular beam with radius \( r \): \( I_y = \frac{\pi}{4} r^4 \)
    - Square beam with width \( b \) and height \( h \): \( I_y = \frac{bh^3}{12} \)
  - Stress
    - Varies linearly in the cross section of the beam
    - \( \sigma = 0 \) in the neutral axis of the beam
    \[
    \sigma = \frac{Mz}{I}
    \]
  - Cantilever beam solution
    - BC’s: \( w(0) = 0, w'(0) = 0, w''(L) = 0, w(0)'' = \text{const.} \)
\[ w_{\text{max}} = w(L) = \frac{FL^3}{3EI} \]
\[ \sigma_{\text{max}} = \sigma(0) = \frac{FLz_{\text{max}}}{l} \]

- Solution to simply supported beam
  - BC's: \( w(0) = w(L) = 0, w''(0) = w''(L) = 0 \)
  - \[ w_{\text{max}} = w \left( \frac{L}{2} \right) = \frac{FL^3}{48EI} \]
  - \[ \sigma_{\text{max}} = \sigma \left( \frac{L}{2} \right) = \frac{FLz_{\text{max}}}{4l} \]

- General solution
  - \( w(x) = \frac{1}{EI} \left( \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4 + \int \int \int q(x) dx^4 \right) \)

**Euler Buckling**
- Elastic instability causing a bifurcation to a lower energy bent state
- Solution to the ODE: \( EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = 0 \)
- General solution is: \( w(x) = A \sin(kx) + B \cos(kx) + Cx + D \), where \( k = \sqrt{\frac{P}{EI}} \)
- The boundary conditions are then used to determine the post-buckled shape.
- The critical load at the lowest mode buckled state can be found to be: \( P_{\text{cr}} = \frac{\pi^2 EI}{(kL)^2} \)

- \( k \) is an “effective length factor”
  - \( k = 1 \) for pinned-pinned boundary
  - \( k = 0.5 \) for fixed-fixed boundary
  - \( k = 0.699 \) for fixed-pinned boundary
  - \( k = 2 \) for fixed-free boundary

- Yielding failure in beams
  - When the axial stress in a beam reaches the yield stress (tension, compression, or bending), it will begin to yield.
Rigidity Theory:

- How do we determine if a pin-jointed structure is rigid?
- “Rigid” means that any deformation of the structure requires an increase in strain energy.
- Maxwell’s Equation
  - Consider a structure with \( j \) joints and \( b \) bars subject to \( k \) kinematic constraints
  - We can say the structure can be rigid if it satisfies the equation:
    \[
    dj - b - k \leq 0 \quad \left\{ \begin{array}{ll}
    d = 2 \text{ in } 2D \\
    d = 3 \text{ in } 3D
    \end{array} \right.
    \]
  - This is a necessary but not sufficient condition.
  - The equation can be generalized to:
    \[
    dj - b - k = m - s
    \]
  - Here, \( m \) represents the number of mechanisms and \( s \) represents the states of self-stress.
  - A mechanism, or inextensional mechanism, means the structure can be moved without the application of stress in the bars.
  - A self-stress means there can be an applied stress in the bars without any corresponding motion of the structure. It also means the structure is ‘statically indeterminate’.
  - Examples:
    - Squares on a hinge
      - 3-bar structure
Equilibrium Matrix Method

- Create a system of equations relating the force at the nodes $f$ to the uniaxial force in the beams $p$ with an equilibrium matrix $A$.

$$f = Ap$$

- $f$ is a vector of length $3j - k$ (3 for each dimension in 3D and $k$ kinematic constraints)
- $p$ is a vector of length $b$
- $A$ is a matrix of size $b \times (3j - k)$
- The equilibrium matrix can be used to determine the number of inextensional mechanisms and states of self-stress.

$$m = b - \text{rank}(A)$$

$$s = (3j - k) - \text{rank}(A^T)$$

- A singular value decomposition (SVD) can be performed on the matrix to find the inextensional mechanisms of the structure.

Mechanics of 2D Structures:

- Square Lattice
  - Uniaxial compression of all the beams when loaded in the $x_1$ and $x_2$ directions.

$\sigma_1$ $\rightarrow$ $\sigma_1$ $\rightarrow$

$\rightarrow$ $\rightarrow$ $\rightarrow$

$t$ $L$

Unit Cell

$\text{Square lattice in compression}$

- The structural stiffness can be found using the rule of mixtures to be:

$$E_1 = \frac{t}{L} E_s$$

- The density of a square lattice is $\bar{\rho} = \frac{2btL}{bL^2} = \frac{2t}{L}$
- Plugging this in, we get

$$\bar{E} = \frac{E_1}{E_s} = \frac{1}{2} \bar{\rho}$$

- Linear scaling of strength and stiffness with density.
- Highly sensitive to imperfections.
- **Honeycomb**
  - Bending dominated structure (2D lattice with depth $b$)

![Hexagonal Lattice](image)

- The strength and stiffness in uniaxial compression are governed by bending of the beams.
- **Analysis for uniaxial compression in x-direction:**

  ![Uniaxial compression in the x_1 direction](image)

  - The load $P$ on the unit cell that arises from the stress is $P = \sigma_1 L (1 + \sin(\theta)) b$
  - The bending moment that arises in the beam can be found to be:

    \[
    M = \frac{PL \sin(\theta)}{2}
    \]

  - From beam theory, the deflection is then:

    \[
    \delta = \frac{PL^3 \sin(\theta)}{12E_s I}
    \]

  - The moment of inertia of a beam is $I = bt^3/12$
  - The deflection of the beam in the $x_1$ direction is $\delta \sin(\theta)$
  - From this, the strain can be found to be:

    \[
    \varepsilon_1 = \frac{\delta \sin(\theta)}{L \cos(\theta)} = \frac{PL^2 \sin^2(\theta)}{12E_s I \cos(\theta)} = \frac{\sigma_1 \sin^2(\theta) (1 + \sin(\theta))}{E_s \cos(\theta)} \left(\frac{L}{E_s I}\right)^3
    \]
The Young’s modulus for the structure is defined as \( E_1 = \sigma_1 / \varepsilon_1 \)

\[
\bar{E} = \frac{E_1}{E_s} = \cos(\theta) \frac{(t/L)^3}{1 + \sin(\theta)} \sin^2(\theta) \left( \frac{t}{L} \right) = \frac{4\sqrt{3}}{3} \left( \frac{t}{L} \right)^3
\]

- The relative density of a hexagon is: \( \bar{\rho} = \frac{2}{\sqrt{3}} \left( \frac{t}{L} \right) \)
- Plugging this in, we get:

\[
\bar{E} = \frac{3}{2} \bar{\rho}^3
\]

**Mechanics of 3D Structures:**

- **Open Cell Foam Model**

  - The density of an open cell foam scales as \( \bar{\rho} \propto \left( \frac{t}{L} \right)^2 \)
  - The area moment inertia of a square beam scales as \( I \propto t^4 \)
  - Because we have a beam in bending, the deflection scales as \( \delta \propto \frac{FL^3}{E_sI} \)
  - Stress scales with applied load as \( \sigma \propto \frac{F}{L^2} \)
  - Strain scales with deflection as \( \varepsilon \propto \frac{\delta}{L} \)
  - Plugging this in for stiffness, we get

\[
E_1 = \frac{\sigma}{\varepsilon} = C \frac{E_s I}{L^4} = C E_s \left( \frac{t}{L} \right)^4
\]

- Using our constituent relationship for relative density, we can say

\[
\bar{E} = \frac{E_1}{E_s} = C_1 \bar{\rho}^2
\]
- **Octet-truss** (stretching-dominated solid)

![Octet-truss structure](image)

**Octet-truss structure**

- The octet-truss is a fully stretching dominated 3D structure, meaning there are no inextensional mechanisms.
- In small strain compression, it is assumed that the beams perpendicular to the applied load carry the stress in tension and allow for deflection of the structure.
- Because it is a uniaxial load that causes the deflection, the stress will scale linearly with relative density, similar to the square lattice and triangular lattice cases.

\[
E = 0.3E_s\bar{\rho}
\]

\[
\sigma_y = 0.3\sigma_{ys}\bar{\rho}
\]

- The 0.3 arises because only \(~1/3\) of the structure (the beams in tension) contributes to the global deflection.
- See: “Effective properties of the octet-truss lattice material”, V.S. Deshpande, N.A. Fleck, M.F. Ashby (2001)
Property Scaling with Relative Density:

- General scaling of strength, stiffness, and fracture toughness with relative density can be defined as

\[ E = B E_s \tilde{\rho}^b \]
\[ \sigma_y = C \sigma_{ys} \tilde{\rho}^c \]
\[ K_{IC} = D \sigma_{TS} \tilde{\rho}^d \sqrt{L} \]

- In 2D, these relations can be defined for different geometries as:

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<th>C</th>
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- In 3D, we instead define scaling as a function of topology and whether the structure is stretching dominated or bending dominated.

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