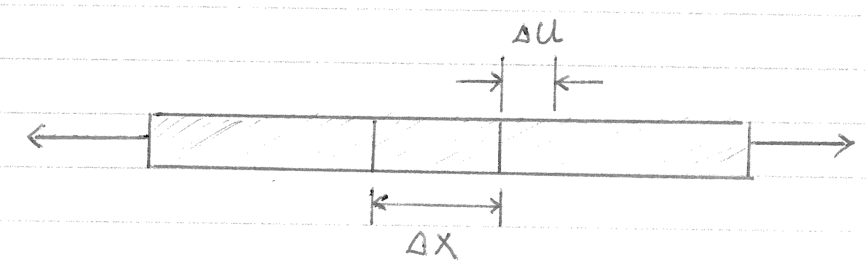
Strain (ε, γ) : measure of deformation.

basic definitions

Consider an element of material stretched in tension.
A section of length ΔX is stretched by an amount Δu

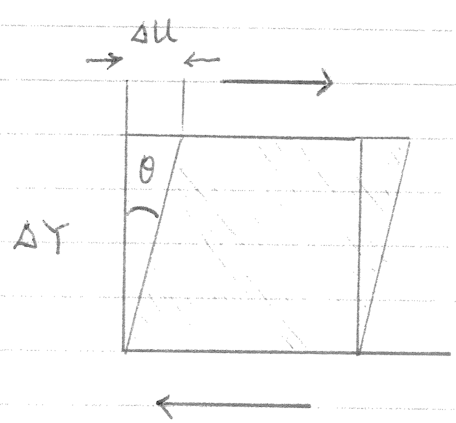


Infinitesimal strain at a point defined as

$$\varepsilon = \lim_{\Delta X \to 0} \left(\frac{\Delta u}{\Delta X} \right) = \frac{du}{dx} \quad \text{(displacement per unit length)}$$

$\hookrightarrow$ axial strain

The basic definition of shear strain is



(for small θ)

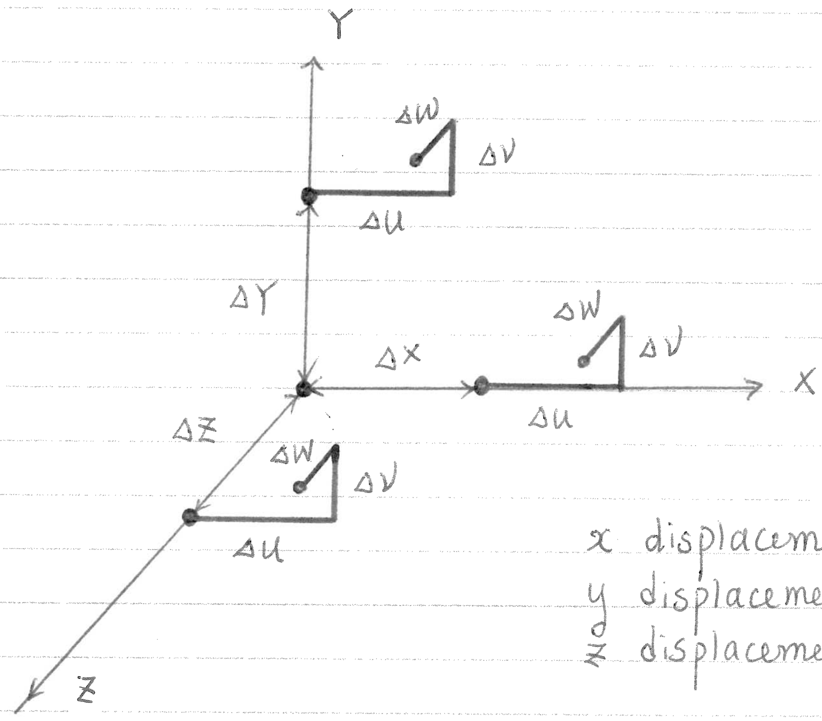
$$\gamma = \lim_{\Delta Y \to 0} \left(\frac{\Delta u}{\Delta Y} \right) = \frac{du}{dy} = \tan\theta \approx \theta$$

(displacement per unit length)

These simple definitions are OK only for very simple deformations
We need a more general statement of strain at a point
to handle complex deformations.

Infinitesimal Strain at a point (3D).

Consider a Cartesian coordinate system imbedded in a solid which is to be deformed. Consider the relative displacements of the material points located on the axes.



x displacement = Δu (Δu_x)
y displacement = Δv (Δu_y)
z displacement = Δw (Δu_z)

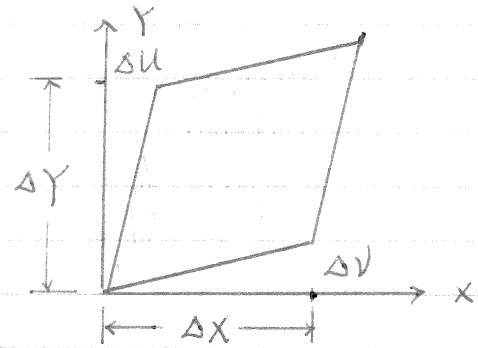
Axial strains :

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} \right) = \frac{\partial u}{\partial x} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \frac{\partial u_z}{\partial z}$$

Shear strains



$$\gamma_{xy} = \lim_{\Delta Y \to 0}\left(\frac{\Delta u}{\Delta Y}\right) + \lim_{\Delta X \to 0}\left(\frac{\Delta v}{\Delta X}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

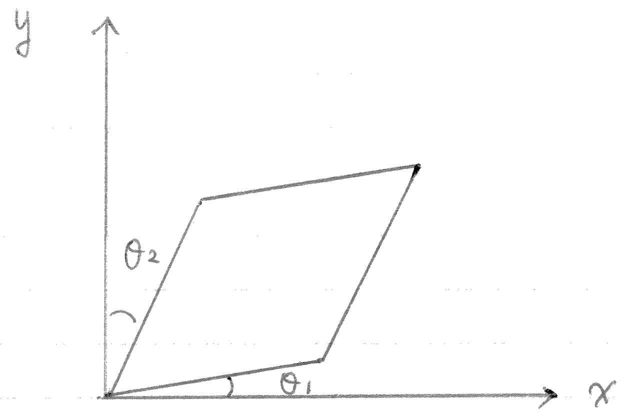
So, strain components can be described by a matrix

$$\text{strain} = \begin{pmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{pmatrix} \quad \text{where } \gamma_{xy} = \gamma_{yx} \text{ etc.}$$

$$\text{strain} = \begin{pmatrix} \varepsilon_1 & \varepsilon_6 & \varepsilon_5 \\ \cdot & \varepsilon_2 & \varepsilon_4 \\ \cdot & \cdot & \varepsilon_3 \end{pmatrix}$$

These are called the engineering strain components.
They are used to define "elastic constants" of crystals.

· why $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$? $= \theta_1 + \theta_2$



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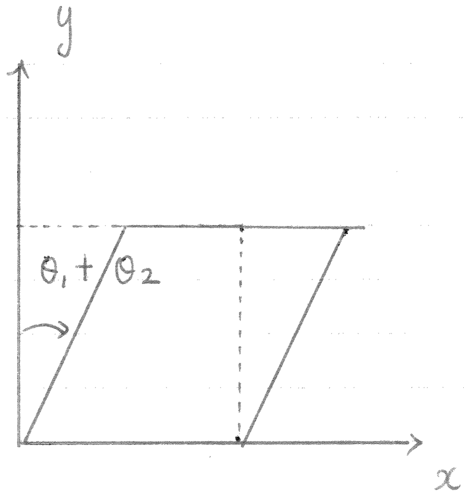

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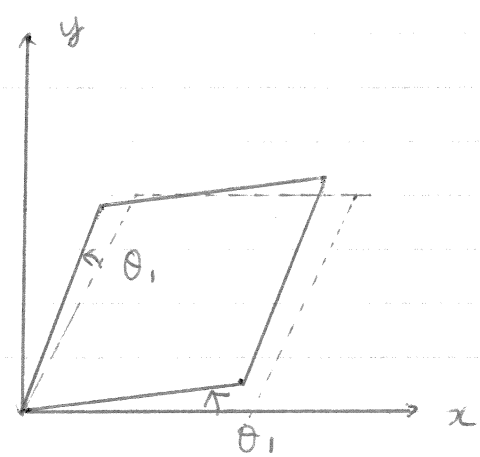


$\gamma_{xy} = \theta_1 + \theta_2$

$\frac{\partial v}{\partial x}$ $\frac{\partial u}{\partial y}$

rigid body rotation

$\omega_z = \theta$.

no contribution to the development of stress !

The Strain Tensor

For some problems in elasticity it is convenient to describe strain as a symmetric 2nd rank tensor.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i, j = x, y, z$$

This gives axial strains as

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \text{etc.}$$

but shear strains are

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

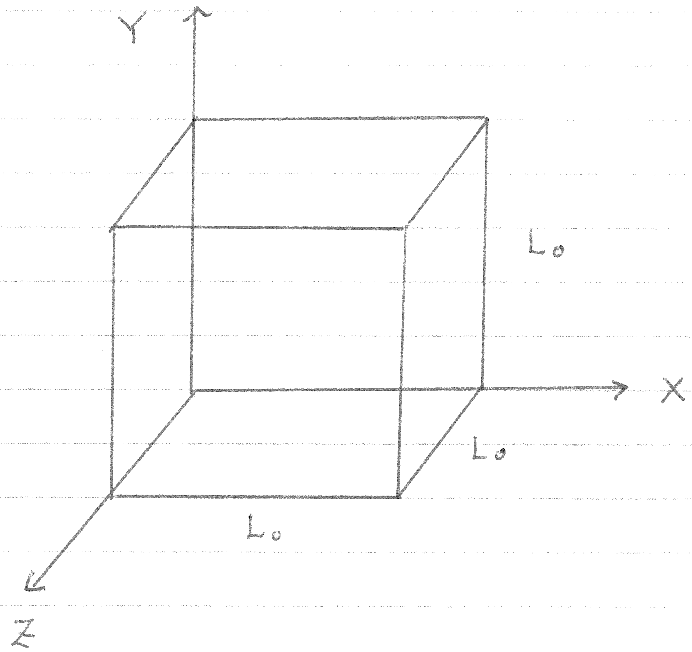
which is different by a factor of 1/2 from above

The strain tensor can also be written as a symmetric matrix

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \qquad \varepsilon_{xy} = \varepsilon_{yx} \quad \text{etc.}$$

Note: transformation to different coordinate systems requires use of strain tensor

Dilatation (volumetric) strain $= \dfrac{\Delta V}{V_o}$



From the definition of strain

$$\left(\because \varepsilon_{xx} = \frac{L_x - L_o}{L_o} \right).$$

$$L_x = L_o (1 + \varepsilon_{xx})$$
$$L_y = L_o (1 + \varepsilon_{yy})$$
$$L_z = L_o (1 + \varepsilon_{zz})$$

Using these we can write

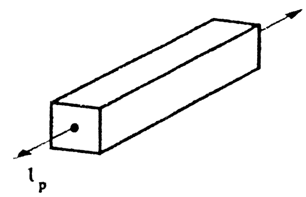
$$\frac{\Delta V}{V_o} = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) - 1$$
$$= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \underbrace{\cdots}_{\text{high order terms}}$$

Ignore the higher order terms (the assumption of small strain).

$$\Delta = \frac{\Delta V}{V_o} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

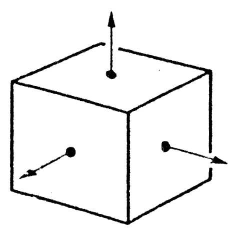
EXAMPLES

Uniaxial Tension (Elastic)



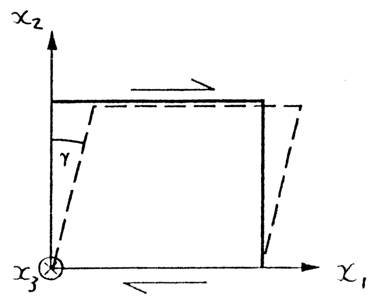
$$\varepsilon_p = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & -\nu\varepsilon_1 & 0 \\ 0 & 0 & -\nu\varepsilon_1 \end{pmatrix} \quad \text{and} \quad \Delta = \varepsilon_1(1-2\nu).$$

Hydrostatic Tension



$$\varepsilon_p = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad \Delta = 3\varepsilon.$$

Engineering Shear



$$e_{21} = \gamma \quad \text{and} \quad e_{12} = 0.$$

Thus

$$\varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}(e_{12} + e_{21}) = \frac{\gamma}{2}.$$